Objectives:

- Find limits of rational functions in cases where we can't substitute
- Find limits of piecewise functions
- Define and use the Squeeze Theorem

We saw last time that if $f(x)$ is a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$. If a is a number not in the domain of $f(x)$, trying to substitute leads to dividing by zero.

1. If trying to plug a into $f(x)$ leads to "$\frac{\text{non-zero}}{0}$", then there is a vertical asymptote at a. This means the one-sided limits can be ∞ or $-\infty$.

   Example $\lim_{x\to 2} \frac{x+5}{x-2}$

   Example $\lim_{x\to 0} \frac{x+1}{x^2}$

2. If trying to plug a into $f(x)$ leads to "$\frac{0}{0}$", the limit is indeterminate. There are a few strategies we can try:

   (a) Factor and Cancel

   Example $\lim_{x\to 2} \frac{x^2-4}{x-2}$

   (b) Combine Fractions

   Example $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

(c) Multiply by the Conjugate

Example $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

Example Let's calculate a limit that can't be approximated numerically on your calculator!

Piecewise Functions

Not all functions are this nice! Piecewise functions require careful thinking about limit definitions:

$$f(x) = \begin{cases} -\sqrt{9+x} & -9 < x < -5 \\ 100 & x = -5 \\ x + 3 & -5 < x \leq 0 \\ x^2 & 0 < x \end{cases}$$

Example 1. $\lim_{x \to -5}$

Example 2. $\lim_{x \to 0} f(x)$:

Example 3. $\lim_{x \to -3} f(x)$:

Don't forget $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$.
Absolute value functions require just as much caution as any other piecewise function.

Example $\lim_{x \to 0} g(x)$ where $g(x) = \dfrac{x}{|x|}$:

The Squeeze Theorem:
If $f(x) \leq g(x)$ for all x near a, then, even if $f(a) > g(a)$, we would expect that:

From this reasonable fact, we can deduce:

The Squeeze Theorem (a.k.a. Sandwich Theorem):

If ____________________ for all x near a (not necessarily for $x = a$),

then ______________________________

The Squeeze Theorem is useful for finding limits of weird functions by "squeezing" them with more cooperative functions:

Example Let $f(x)$ be a mystery function. The only thing we know about f is
$3 - 2x - x^2 \leq f(x) \leq -2x + 3$ for all $x \neq a$. Find $\lim_{x \to 0} f(x)$.

Example Find $\lim_{t \to 0} t^2 \sin\left(\frac{1}{t}\right)$.